Z-Transform Implementation of the Perfectly Matched Layer for Truncating FDTD Domains

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***Abstract*—A simple algorithm for implementing the perfectly matched layer (PML) is presented for truncating finite difference time domain (FDTD) computational domains. The algorithm is based on incorporating the Z-transform method into the PML FDTD implementation. The main advantage of the algorithm is its simplicity as it allows direct FDTD implementation of Maxwell's equations in the PML region. In addition, the formulations are independent of the material properties of the FDTD computational domain. Numerical examples are included to demonstrate the effectiveness of these formulations.**

***Index Terms*—Finite difference time domain, perfectly matched layer, Z-transform.**

I. Introduction

THE finite difference time domain (FDTD) method [1] has been widely used for solving electromagnetic problems. Recently, the Z-transform method has been incorporated into the FDTD simulations of frequency dependent media [2], [3]. This method allows direct discretizations of Maxwell's equations in the time domain and also allows the use of signal processing theory in the FDTD research.

An important issue for solving open region problems using the FDTD method is the truncation of the computational domain [1]. Berenger's split-field perfectly matched layer (PML) [4] has been shown to be an effective FDTD absorbing boundary conditions. Different interpretations of Berenger's PML based on the stretched coordinate PML [5] or the anisotropic PML [6] approaches have been introduced. Recently, a novel PML formulation, based on combining the convolution theorem and the stretched coordinate PML, has been proposed for truncating general FDTD domains [7]. A recursive convolution method similar to [8] has been used to discretize and compute the corresponding convolution terms. In [9], the auxiliary differential equation method has been used for implementing the stretched coordinate PML without the need for computing the convolutional terms introduced in [7].

In this paper, an alternative technique using the Z-transform method is incorporated into the FDTD implementation of the stretched coordinate PML. The main advantage of this method is its simplicity, as it allows direct FDTD implementation of the Maxwell's equations in the PML region. In addition, the formulations are implemented using $\mathbf{D}$ and $\mathbf{B}$ fields rather than the conventional $\mathbf{E}$ and $\mathbf{H}$ formulations. This allows the PML to be independent of the material properties of the FDTD computational domains [10] and can be used for truncating general lossy, dispersive, anisotropic or nonlinear media. Numerical tests have been carried out in two-dimensional domains to validate the proposed method.

II. Formulation

Using the coordinate stretching approach [5], the frequency domain modified Maxwell's equations in the PML region can be written as

$$\nabla_s \times \mathbf{H} = j\omega \mathbf{D} \tag{1}$$

$$\nabla_s \times \mathbf{E} = -j\omega \mathbf{B} \tag{2}$$

where the operator ∇_s is expressed as

$$\nabla_s = \widehat{a}_x S_x^{-1}\partial_x + \widehat{a}_y S_y^{-1}\partial_y + \widehat{a}_z S_z^{-1}\partial_z \tag{3}$$

with ∂_x, ∂_y, and ∂_z are the partial derivatives with respect to x, y, and z, and $S_\eta\,(\eta = x, y, z)$ are called the complex stretched coordinate variables defined as [5]

$$S_\eta = 1 + \frac{\sigma_\eta}{j\omega\varepsilon_o} \tag{4}$$

where σ_η is the conductivity profile along the η-direction in the PML region. The $\mathbf{D}$ and $\mathbf{B}$ fields in (1) and (2) are given by the following relations:

$$\mathbf{D} = \varepsilon_o \widehat{\varepsilon}_r(\omega)\mathbf{E}; \qquad \mathbf{B} = \mu_o \widehat{\mu}_r(\omega)\mathbf{H}. \tag{5}$$

where $\widehat{\varepsilon}_r(\omega)$ and $\widehat{\mu}_r(\omega)$ are, respectively, the relative permittivity and permeability of the FDTD computational domain. The advantage of using $\mathbf{D}$ and $\mathbf{B}$ in the above formulations is that the PML becomes independent of the material properties ($\widehat{\varepsilon}_r(\omega)$, $\widehat{\mu}_r(\omega)$) of the FDTD computational domain [10]. Hence, (1) and (2) can be used for truncating general domains, such as lossy, dispersive, anisotropic or nonlinear, without modifications and all that is needed is to modify (5) accordingly. Different techniques are available for discretizing (5) [2], [3], [11]. To discretize (1) and (2), consider, as an example, the D_z-field component of (1)

$$j\omega D_z = S_x^{-1}\partial_x H_y - S_y^{-1}\partial_y H_x. \tag{6}$$

Due to the frequency dependence of $S_\eta\,(\eta = x, y)$, the transformation of (6) to the time domain will result in convolutions on the right hand side [7] as

$$\partial_t D_z = \mathbf{S}_x(t) * \partial_x H_y - \mathbf{S}_y(\mathbf{t}) * \partial_y H_x \tag{7}$$

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where ∂_t is the partial derivative with respect to time, $\mathbf{S}_\eta(t)$, ($\eta = x, y$), is the inverse Fourier transform of S_η^{-1} and $*$ represents the convolution operation. As the convolution in the time domain is just a multiplication in the Z-domain [14], it is more efficient to transform (7), firstly, into the Z–domain and then the corresponding FDTD implementation can be obtained directly. Hence, (7) can be written in the Z-domain as

$$\left(1 - z^{-1}\right) \frac{D_z}{\Delta t} = \mathbf{S}_x(z)\partial_x H_y - \mathbf{S}_y(z)\partial_y H_x \qquad (8)$$

where Δt is the time step and $\mathbf{S}_\eta(z)$, ($\eta = x, y$), is the Z-transform of $\mathbf{S}_\eta(t)$ which can be obtained easily from S_η^{-1} as

$$\mathbf{S}_\eta(z) = \frac{1 - z^{-1}}{1 - z^{-1}e^{-\sigma_\eta \Delta t/\varepsilon_o}}. \qquad (9)$$

Substituting (9) into (8), we obtain

$$D_z\left(1 - z^{-1}\right) = \frac{\Delta t\left(1 - z^{-1}\right)}{1 - z^{-1}e^{-\sigma_x \Delta t/\varepsilon_o}}\partial_x H_y - \frac{\Delta t\left(1 - z^{-1}\right)}{1 - z^{-1}e^{-\sigma_y \Delta t/\varepsilon_o}}\partial_y H_x. \qquad (10)$$

Introducing the following auxiliary variables:

$$f_{zx} = \frac{\Delta t}{1 - z^{-1}e^{-\sigma_x \Delta t/\varepsilon_o}}\partial_x H_y = e^{-\sigma_x \Delta t/\varepsilon_o} z^{-1} f_{zx} + \Delta t \partial_x H_y \qquad (11)$$

and

$$f_{zy} = \frac{\Delta t}{1 - z^{-1}e^{-\sigma_y \Delta t/\varepsilon_o}}\partial_y H_x = e^{-\sigma_y \Delta t/\varepsilon_o} z^{-1} f_{zy} + \Delta t \partial_y H_x \qquad (12)$$

(10) can be written as

$$D_z\left(1 - z^{-1}\right) = f_{zx} - z^{-1} f_{zx} - \left(f_{zy} - z^{-1} f_{zy}\right). \qquad (13)$$

Substituting (11) and (12) instead of f_{zx} and f_{zy} in (13)f

$$D_z\left(1 - z^{-1}\right) = \Delta t\left(\partial_x H_y - \partial_y H_x\right) + \left(e^{-\sigma_x \Delta t/\varepsilon_o} - 1\right) z^{-1} f_{zx} - \left(e^{-\sigma_y \Delta t/\varepsilon_o} - 1\right) z^{-1} f_{zy}. \qquad (14)$$

Discretizing the space derivatives in (14) following Yee's algorithm [1] and using the fact that the z^{-1} operator corresponds to a delay of one time step in the discrete time domain [14], (14) can be written in FDTD form as

$$D^{n+1}_{z_{i,j,k+1/2}} = D^{n}_{z_{i,j,k+1/2}} + \frac{\Delta t\left(H^{n+1/2}_{y_{i+1/2,j,k+1/2}} - H^{n+1/2}_{y_{i-1/2,j,k+1/2}}\right)}{\Delta x + g_{zx1}(i) f^{n}_{zx_{i,j,k+1/2}}} - \frac{\Delta t\left(H^{n+1/2}_{x_{i,j+1/2,k+1/2}} - H^{n+1/2}_{x_{i,j-1/2,k+1/2}}\right)}{\Delta y - g_{zy1}(j) f^{n}_{zy_{i,j,k+1/2}}} \qquad (15)$$

where $g_{zx1}(i) = e^{-\sigma_x(i)\Delta t/\varepsilon_o} - 1$, $g_{zy1}(j) = e^{-\sigma_y(j)\Delta t/\varepsilon_o} - 1$ and Δx and Δy are the space cell size in the x and y directions, respectively, and the auxiliary variables $f^{n}_{zx_{i,j,k+1/2}}$ and $f^{n}_{zy_{i,j,k+1/2}}$ are obtained easily from (11) and (12) as

$$f^{n+1}_{zx_{i,j,k+1/2}} = g_{zx2}(i)^{n}_{zx_{i,j,k+1/2}} + \Delta t A \qquad (16)$$

$$f^{n+1}_{zy_{i,j,k+1/2}} = g_{zy2}(j) f^{n}_{zy_{i,j,k+1/2}} + \Delta t B \qquad (17)$$

where $g_{zx2}(i) = e^{-\sigma_x(i)\Delta t/\varepsilon_o}$ and $g_{zy2}(j) = e^{-\sigma_y(j)\Delta t/\varepsilon_o}$. Similar formulations can be obtained for the other D and B field components. To complete the FDTD iterative cycle for computing E and H field components, the proposed formulations should be combined with the FDTD implementation of (5), which can be obtained for different media by using techniques similar to those described in [2], [3], [11]–[13].

The auxiliary variables introduced above are stored only in the PML regions. This is because $\sigma_\eta(\eta = x, y, z)$ are zero within the FDTD domains. The proposed formulations are applied only in the corner PML regions and simpler formulations can be obtained in the face and edge regions [4], [5]. It should be noted that the proposed formulations require the same number of additional auxiliary variables, and hence the same numerical cost, as the formulations in [7] and [9], but it has the advantage of the simplicity in discretizing (1) and (2) in the PML region. Finally, the proposed formulations can be extended for the general expression of $S_\eta(\eta = x, y, z)$ [7].

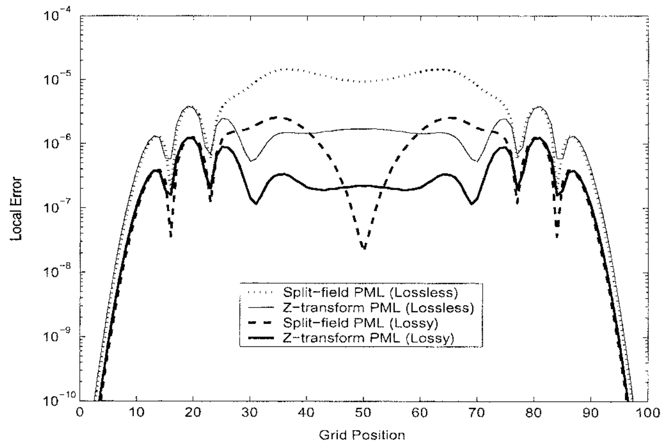

Fig. 1. Local error for the PML/computational domain interface along the line $(x, -25\Delta y)$ as observed at time $100\Delta t$ using the Z-transform PML and the split-field PML formulations for PML [8, 2, 0.1%] and lossless ($\varepsilon_r = \mu_r = 1$, $\sigma = 0.0$) and lossy ($\varepsilon_r = \mu_r = 1$, $\sigma = 0.01$) FDTD domains.

III. Numerical Study

In order to validate the proposed method, a numerical study has been carried out in a 2-D FDTD domain for the TM case. In the following tests, the computational domain was chosen to be $100\Delta \times 50\Delta$ cells, where $\Delta = \Delta x = \Delta y$ is the space cell size in the x and y directions, and excited by a point source at its center. The reference FDTD solution is calculated using a larger computational domain ($1400\Delta \times 1400\Delta$).

The performance of the proposed Z-transform PML was first compared with the split-field PML formulations for lossless ($\varepsilon_r = \mu_r = 1$, $\sigma = 0.0$) and for lossy ($\varepsilon_r = \mu_r = 1$, $\sigma = 0.01$) FDTD domains terminated by PML [8, 2, 0.1%] layer, as defined in [4]. The excitation used in this test, was the derivative of the pulse used in [15]. This pulse was preferred as it has low numerical grid dispersion [16]. The time step used was $\Delta t = 25$ ps and the space cell size was 15 mm. In this test, the discretization of (5) can be obtained in a straightforward manner [1]. Figs. 1 and 2 show the local and the global errors, as defined in [15], for both cases. The local error was calculated for the PML/computational domain interface along the line $(x, -25\Delta y)$ as observed at time $100\Delta t$. Two sets of results are presented: one for the Z-transform PML, and one for

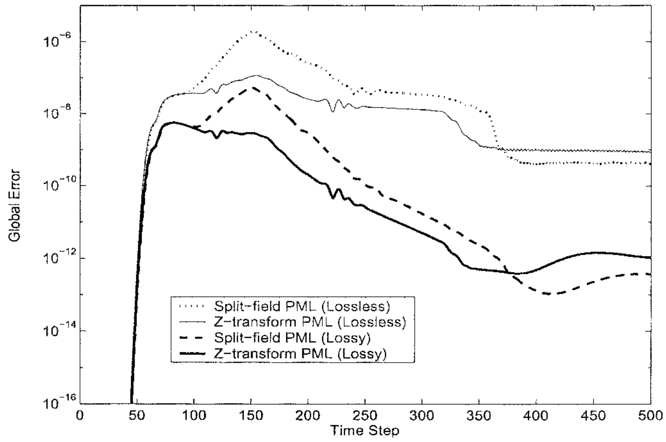

Fig. 2. Global error in the computational domain using the Z-transform PML and the split-field PML formulations for PML [8, 2, 0.1%] and lossless ($\varepsilon_r = \mu_r = 1, \sigma = 0.0$) and lossy ($\varepsilon_r = \mu_r = 1, \sigma = 0.01$) FDTD domains.

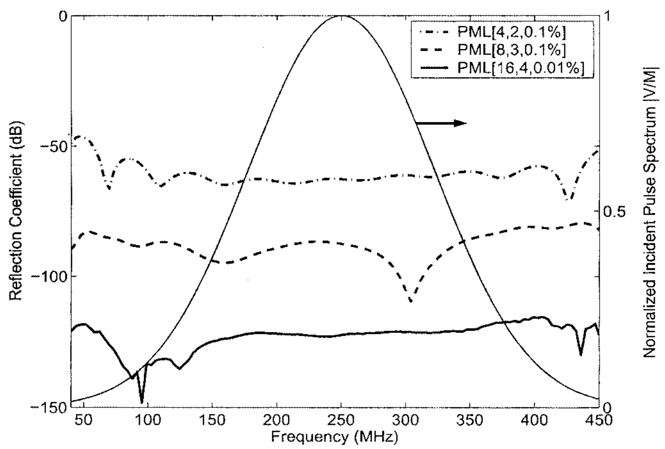

Fig. 3. Reflection coefficient for different PML layers as observed one cell from the PML/computational domain interface at the point ($-49\Delta x$, 0) for lossy dispersive media.

the split field PML formulations. In the latter case, each D and B field component in (1) and (2) is splitted into two subcomponents [4]. It can be observed from Figs. 1 and 2 that, for the given PML parameters, the Z-transform PML gives better absorbing performance compared with the split field PML formulations.

The performance of the proposed formulations was also tested for nonmagnetic lossy dispersive FDTD domain with a dielectric property of $\widehat{\varepsilon}_r(\omega) = \varepsilon_\infty + (\varepsilon_s - \varepsilon_\infty)/(1 + j\omega t_0) + \sigma/j\omega\varepsilon_o$ where ε_∞ is the infinite frequency permittivity, ε_s is the static permittivity, t_0 is the relaxation time and σ is the conductivity of the medium. This model is used to approximate the muscle tissue in the frequency range 40–433 MHz [11] with the parameters $\varepsilon_\infty = 15$, $\varepsilon_s = 120$, $t_0 = 6.67$ ns, and $\sigma = 0.64$ S/ m. The space cell size was 10 mm and the time step was 16.67 ps. Fig. 3 shows the reflection coefficient for PML [4, 2, 0.1%], PML [8, 3,0.1%] and PML [16, 4, 0.01%] layers as observed one space cell from the PML/computational domain interface at the point ($-49\Delta x$, 0). The reflection coefficient was calculated as $R(\text{dB}) = 20\log_{10}\left|F\left\{E_z^R(t) - E_z^T(t)\right\}/F\left\{E_z^R(t)\right\}\right|$ where $F\{.\}$ is the Fourier transform operation, $E_z^R(t)$ and $E_z^T(t)$ are, respectively, the reference and the calculated FDTD solutions. The spectrum of the excitation pulse used in this test is also shown in Fig. 3. As it can be seen from Fig. 3, the Z-transform PML provides good absorbing performance for all PML layers over the frequency band of interest. It must be noted that the discretization of (5) in this test was carried using the Z-transform theory described in [2].

IV. Conclusion

A simple method is presented for FDTD implementation of the stretched coordinate PML formulations using the Z-transform technique. The method has the advantage of the simplicity; the FDTD implementation of the Maxwell's equations in the PML region is obtained in a very simple manner. In addition, the formulations are independent of the material properties of the FDTD domain, and hence general lossy, dispersive, anisotropic and nonlinear media can be truncated easily. Numerical tests show that the proposed Z-transform PML formulations give good absorbing performance for lossless, lossy and dispersive FDTD domains.

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